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.

M $T : M \to M$. $f : M \to R_1$ $-$. $T^n : M \to M$ $f(T^n x)$.

$$\int_M f \overset{\text{def}}{=} \frac{1}{m} \sum_{x \in M} f(x), \quad |M| = m.$$

, , $T^m = Id = E -$. x_0 .

1.

$$\frac{1}{m} \sum_{l=1}^{m} f\big(T^l(x_0)\big) = \lim_{l \to \infty} \frac{1}{l} \sum_{p=1}^{l} f\big(T^p(x_0)\big) = \int_M f \overset{\text{def}}{=} \frac{1}{m} \sum_{x \in M} f(x) \qquad (1)$$

:

, $l > m$. $k > 0$, $k \in N$, $l = km + r(l)$, $0 \le r(l) < m$ l. (1) :

$\frac{1}{l} \sum_{p=1}^{l} f\big(T^p(x_0)\big) = \frac{1}{km+r(l)} \sum_{p=1}^{km+r(l)} f\big(T^p(x_0)\big) = \frac{1}{km+r(l)} \big[\sum_{p=1}^{km} f(x_p) + \sum_{i=1}^{r(l)} f(x_i)\big] = \frac{1}{km+r(l)} \big[k \sum_{i=1}^{m} f(x_i) + \sum_{i=1}^{r(l)} f(x_i)\big] = k \frac{\sum_{i=1}^{m} f(x_i)}{km+r(l)} + \frac{\sum_{i=1}^{r(l)} f(x_i)}{km+r(l)} = k \frac{\mu}{km+r(l)} + \frac{\sum_{i=1}^{r(l)} f(x_i)}{km+r(l)} = \mu \frac{k}{km+r(l)} + \frac{\sum_{i=1}^{r(l)} f(x_i)}{km+r(l)} = \mu \frac{1}{m+r(l)/k} + \frac{\sum_{i=1}^{r(l)} f(x_i)}{km+r(l)} \to \frac{\mu}{m}$

T, $u, T = T(x,u)$, , u $U(x)$. , $\mathcal{U} = \mathcal{U}(x)$ M ( ) (). $\mathcal{U}$:

$$\frac{1}{m} \sum_{l=0}^{m-1} f\big(T^l(x, U_l)\big) = \int_M f = \frac{1}{m} \sum_{l=0}^{m-1} f\big(T^l(x_0, \mathcal{U})\big) = \lim \frac{1}{m} \sum_{l=0}^{\infty} f\big(T^l(x_0, \mathcal{U})\big) \qquad (**)$$

$$\sum_{l=0}^{m-1} f\big(T(x, U_l)\big)$$

. $x = x_0$ $u_0 = \mathcal{U}(x_0)$ x_0 $x_1 = T(x_0, u(x_0))$, x_1 $u_1 = \mathcal{U}(x_1)$ x_1 $x_2 = T(x_1, u(x_1))$. x_0, , (**).

, $T = t(x, u, v)$, $u(x)$, $v(x)$ $U(x)$, $V(x)$.

(), $u(x)$, $v(x)$, : - $k \le m$ $\frac{1}{k} \sum_{l=1}^{k} f\big(T(x_l, u_l v_l)\big)$, - .

, m , - $u(x)$, $v(x)$.

$\frac{1}{m} \sum_{l=1}^{m} f\big(T(x_l, u_l v_l)\big) = \int_M f = \frac{1}{m} \sum_{x \in M} f(x) =$

$= \max_{\{\mathcal{U}\}} \min_{\{\mathcal{V}\}} \frac{1}{m} \sum_{l=1}^{m} f\big(T(x_l, u_l v_l)\big) =$

$= \min_{\{\mathcal{V}\}} \max_{\{\mathcal{U}\}} \frac{1}{m} \sum_{l=1}^{m} f\big(T(x_l, u_l v_l)\big)$

$\mathcal{U}, \mathcal{V}$ I II $m-$ ( ).

.

$k < m$.

n .

T , $x \in M$ n $u_1(n)...u_n(x)$, $U_1(x)...U_n(x)$, $T = T(x, u_1...u_n) = T(x, u)$, , u_i $x \in M$ - T M.

M n $f_1...f_n$, $u_i(x)$ m .

, ,

$$\frac{1}{m}\sum_{l=1}^{m} f_i\big(T(x_l, u^l)\big) = \int\limits_M f_i = \frac{1}{m}\sum_{x\in M} f_i(x)$$

$u^l = (u_1^l, ...u_n^l)$.

$$\mathrm{Val}_{\{\mathcal{U}_1\}...\{\mathcal{U}_n\}} \frac{1}{m}\sum_{l=1}^{m} f\big(T(x_l, u^l)\big) = \int\limits_M f = \frac{1}{m}\sum_{x\in M} f_i(x)$$

$f = (f_1...f_n)$, $\mathrm{Val}_{\{\mathcal{U}_1\}...\{\mathcal{U}_n\}}$ $-$ n $\mathcal{U}_1, ...\mathcal{U}_n$ $m-$ $\frac{1}{m}\sum_{l=1}^{m} f\big(T(x_l, u^l)\big)$.

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$(M, \Sigma, \mu)-$ μ, $A \in \Sigma$, $\mu A = 0$, $B \subset A$ Σ. $\{B_\alpha\}$, $B_\alpha \in \Sigma$ $F(\{B_\alpha\})$, B_α.

, $\mathcal{B} = \{b_i, i \in I\}$ M, . -, $A \in \Sigma$ $C \in F(\mathcal{B})$, $A \subset C$, $\mu(C \setminus A) = 0$, -, $x_1, x_2 \in M$, $x_1 \neq x_2$ $i \in I$, $x_1 \in B_i$, $x_2 \notin B_i$ $x_2 \in B_i$, $x_1 \notin B_i$.

$\mathcal{B} = \{B_i\}$ $e_i = \pm 1$. $B_i^{(e_i)} = B_i$, $e_i = 1$, $B_i^{(e_i)} = M \setminus B_i$, $e_i = -1$.

$\{e_i, i \in I\}$ $\bigcap_{i\in I} B_i^{e_i}$. .

(M, Σ, μ) $\mathcal{B}$, $\bigcap_{i\in I} B_i^{e_i}$.

, (M, Σ, μ) (mod0) $\mathcal{B}$, M 1 $(\bar{M}, \bar{\Sigma}, \bar{\mu})$ $\bar{\mathcal{B}} = \{\bar{B}_i, i \in I\}$, $\bar{B}_i \bigcap M = B_i$ $i \in I$.

, (mod0) , (mod0) .

(M, Σ, μ) (mod0) .

, , $\sigma-$, . .

(M, Σ, μ) $\xi = \{C\}$ C, , $\bigcup_{C\in\xi} = M$. $\bigcup_{C\in\xi} = M$ (mod0), ξ, (mod0).

$A \in \Sigma$, $C_\xi \in \xi$ ξ.

ξ , $\mathcal{B} = \{B_i, i \in I\}$ ξ $C_1, C_2 \in \xi$ $i \in I$, $C_1 \in B_i$, $C_2 \notin B_i$ $C_2 \in B_i$, $C_1 \notin B_i$.

, - $M \setminus \xi$ M ξ .

(M, Σ, μ) - σ Σ μ , $T : M \to M$. $\{\xi_n\}$ M $\{T_n\}$, T_n ξ_n, ξ_n ξ_n, $C_j^{(n)}$, $j_n = 1, ..., q_n$. $\Sigma(\xi_n)$ $\sigma-$ M, (mod0) ξ_n, ξ_0 , M. $\xi_n \to \xi_0$, $A \in \Sigma$ $A_n \in \Sigma(\xi_n)$, $\mu(A_n \Delta A) \to 0$.

$|\xi_n|-$ ξ , C_j^n $r_j \le q_n$ $T_n^{r_j} C_j^{(n)} = C_j^{(n)}$, $C_j^{(n)}$. , $T_n^{r_j} x = x$ $x \in C_j^{(n)}$, p_n , $T_n^{p_n} = \mathrm{Id}$.

. (.)

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$T-$ (M, Σ, μ). $x \in M$ T, $n \in Z = \{1, 2, ...\}$ $T^n x = x$.

T , $\mathcal{P}(T, M)$ - $\mu \mathcal{P}(T, M) = 0$.

$\mathcal{L}$.

1. $T-$ (M, Σ, μ). $\varepsilon > 0$ $n \in Z$ $E \in \Sigma$, $E, TE, ..., T^{n-1}E$, ,

$$\mu\Big(\bigcup_{i=0}^{n-1} T^i E\Big) > 1 - \varepsilon.$$

. .

1. $n \in Z$ $F_n \in \Sigma$, $\mu(F_n) > 0$, $F_n, TF_n, ...$ $T^{n-1}F_n$.

n. $n = 1$. n, , $n+1$.

, F_n $F_n^{'} \in \Sigma$, $F_n^{'} \subset F_n$, $\mu(F_n^{'} \Delta T^n F_n^{'}) > 0$. , $A \in \Sigma$, $A \subset F_n$ $\mu(A \Delta T^n(A)) = 0$. $M \in \mathcal{L}$, , $T^n\big|_{F_n} = \mathrm{Id}\ (\mathrm{mod} 0)$, T. $F_{n+1} = F_n^{'} \setminus T^n F_n^{'}$. $F_{n+1}, TF_{n+1}, ..., T^n F_{n+1}$ $\mu(F_{n+1}) > 0$.

2. $A, B \in \Sigma$, , $\mu(A \Delta B) = 0$. F_n .1 $\{F_n^\alpha\}$–$F_n = \bigcup_\alpha F_n^\alpha$. , α $F_n \in \gamma$.

$\hat{F_n} \in \Sigma$, , $F_n \in \Sigma$, $T^i F_n$ $T^j F_n$, $0 \le i < j \le n-1$ $\hat{F_n} \subset F_n$ $\hat{F_n}$. $T^i \hat{F_n}$ $T^j \hat{F_n}$ $0 \le i < j \le n-1$.

3. $\hat{F_m}$ m. $x \in \hat{F_m}$

$$r(x) = \min\{r \ge 1 : T^r x \in \hat{F_m}\}$$

, $m \le r(x) \le 2m - 1$ x. $\hat{F_m}$.

$$G = \{x \in \hat{F_m} | r(x) \ge 2m\}$$

$$\hat{\hat{F_m}} = \hat{F_m} \bigcup T^m G.$$

$T^i \hat{\hat{F_m}}$ $T^j \hat{\hat{F_m}}$, $0 \le i < j \le m-1$ $\mu G > 0$, $\mu \hat{\hat{F_m}} > \mu \hat{F_m}$, $\hat{F_m}$. $\mu G = 0$.

$$\hat{F_{m,k}} = \{x \in \hat{F_m} | r(x) = k\}.$$

$$\bigcup_{k=m}^{2m-1} \hat{F_{m,k}} = \hat{F_m}.$$

4.

$$M_1 = \bigcup_{k=m}^{2m-1} \bigcup_{i=0}^{k-1} T^i \hat{F_{m,k}}.$$

, $M_1 = M$. , M_1 . $x \in T^i F_{m,k}^{\hat{}}$ $i < k-1$, $Tx \in T^{i+1} F_{m,k}^{\hat{}}$. $x \in T^{k-1} F_{m,k}^{\hat{}}$ $Tx \in \hat{F_m} = \bigcup_{k=m}^{2m-1} F_{m,k}^{\hat{}} \subset M$.

$\mu(M \setminus M_1) > 0$, .1 $F_m \subset M \setminus M_1, \mu F_m > 0$ $F_m, TF_m, \dots T^{m-1} F_m$. $\hat{F_m} \bigcup F_m$, $\hat{F_m}$, .

5. m $n/m < \varepsilon$

$$E = \bigcup_{k=m}^{2m-1} \bigcup_{i=0(modn), 0 \le i \le k-1} T^i F_{m,k}^{\hat{}}.$$

, E– . , $E, TE \dots T^{n-1} E$.

$$\bigcup_{i=0}^{n-1} T^i E = \bigcup_{k=m}^{2m-1} \bigcup_{i=0}^{p_k} T^i F_{m,k}^{\hat{}},$$

$$p_k = \max\{i | 0 \le i \le k-1, i = n-1(modn)\}.$$

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$$\bigcup_{k=m}^{2m-1} \bigcup_{i=0}^{k-1} T^i F_{m,k}^{\hat{}} = M(mod0).$$

$k, m \le k \le 2m-1$, .

$$\frac{\mu(\bigcup_{i=0}^{p_k} T^i F_{m,k}^{\hat{}})}{\mu(\bigcup_{i=0}^{k-1} T^i F_{m,k}^{\hat{}})} = \frac{p_k + 1}{k} \ge \frac{k-n}{k} = 1 - \frac{n}{k} \ge 1 - \frac{n}{m} > 1 - \varepsilon.$$

,

$$\mu(\bigcup_{i=1}^{n-1} T^i E) = \frac{\mu(\bigcup_{k=m}^{2m-1} \bigcup_{i=0}^{p_k} T^i F_{m,k}^{\hat{}})}{\mu(\bigcup_{k=m}^{2m-1} \bigcup_{i=0}^{k-1} T^i F_{m,k}^{\hat{}})} = \frac{\sum_{k=m}^{2m-1} \mu(\bigcup_{i=0}^{p_k} T^i F_{m,k}^{\hat{}})}{\sum_{k=m}^{2m-1} \mu(\bigcup_{i=0}^{k-1} T^i F_{m,k}^{\hat{}})} > 1 - \varepsilon.$$

. E ,

$$\bigcup_{i=0}^{2n-1} T^i E = M.$$

. $g(h)$. $T : M \to M$ $g(h)$, $\xi_n \to \varepsilon$ T_n, ξ_n ,

$$\sum \mu(T(C_i^n) \Delta T_n(C_i^n)) < g(q_n), n = 1, 2 \dots$$

, T $f(n)$ T T_n ξ_n.

2. $E \in \Sigma$ $k \in Z = \{0, 1, 2 \dots\}$

$$\mu(T^k E \Delta T_n^k E) \le \sum_{i=0}^{k-1} \mu(T(T_n^i E) \Delta T_n^{i+1} E).$$

:

$\mu(T^k E \Delta T_n^k) \le \mu(T^k E \Delta T^{k-1} T_n E) + \mu(T^{k-1} T_n E \Delta T^{k-2} T_n^2 E) + \dots$
$+\mu(T T_n^{k-1} \Delta T_n^k E) = \sum_{i=0}^{k-1} \mu(T(T_n^i E \Delta T_n^{i+1} E)$.

, .

. T $f(n) = a_n | \ln n$, $\{a_n\}$– - .

: T.

n $A_n \in \Sigma$ $T^k A_n$ $0 \le k \le n-1$ $\mu(\bigcup_{k=0}^{n-1} T^k A_n) > 1 - \frac{1}{n}$. T_n :
$T_n x = Tx$ $x \in \bigcup_{k=0}^{n-2} T^k A_n$
$T_n x = T^{-n+1} x$ $x \in T^{n-1} A_n$
$T_n x = x$ $x \in M \setminus \bigcup_{k=0}^{n-1} T^k A_n$.

ξ_n. n $\theta_n \to \varepsilon$, k_n θ_n , $k_n > 8 \ln k_n$ $2 \le k_n < \min(a_n, n)$. ξ_n, $M \setminus A_n$

$$A_n, C_{r_0}, T^{-1} C_{r_1}, \dots T^{-n+1} C_{r_{n-1}},$$

$C_{r_i}, 0 \le i \le n-1$ θ_n. $T^p \xi_n$ $0 \le p \le n-1$, $T^p A_n$, ξ A_n, $M \setminus T^p A_n$ .. $M \setminus T^p A_n$ $T^p \xi_n$. $T^p A_n$ ξ_n θ_n.

$V_{k=0}^{n-1} T^k \xi_n$ ξ_p $T^p A_n$ $0 \le p \le n-1$ $M \setminus \bigcup_{p=0}^{n-1} T^p A_n$. $\xi_n = V_{k=0}^{n-1} T^k \xi_n$. $M \setminus \bigcup_{k=0}^{n-1} T^k A_n$ ξ_n θ. , $\xi_n \to \varepsilon$. T_n , $T_n \xi_n = \xi_n$.

q_n ξ_n $q_n \le n k_n^n + 1$. $n \le 2^n - 1 \le k_n^n - 1$

$$q_n \le n(k_n^n + 1) \le (k_n^n - 1)(k_n^n + 1) \le k_n^{2n}.$$

$\frac{1}{2n} \le \frac{\ln k_n}{\ln q_n}$.

$$\sum_{i=1}^{q_n} \mu(T C_i^{(n)} \Delta T_n C_i^{(n)}) \le 2\mu(M \setminus \bigcup_{k=0}^{n-2} T^k A_n) \le \frac{4}{n} \le 8 \frac{\ln k_n}{\ln q_n} < \frac{k_n}{\ln q_n} < \frac{a_n}{\ln q_n}$$

.

. Γ :

$$< I = \{i\}, (M, \Sigma, \mu), \{U_i\} = U_i, T(x,u), f = \{f_i\}_{i \in I}, S_i = \{\lambda_i\}, S, P, F > .$$

I– (), (M, Σ, μ)– . U_i– i ($x \in M$). $T(x,u)$– M, u, F– M. S_i– i. S– Γ. $P : S \to F$–, $s \in S$ T M. f_i– i; $f_i : M \to R_1$.

Γ . $i \in I$ ε_i s_i , $\varepsilon = \inf_{i \in I} \varepsilon_i > 0$ $s = \{s_i\} \in S$. P $P(s) = T \in F$. T_ε, T ε. i Γ $\psi_i = (\varepsilon_i, s_i)$. " "– C_j^ε ξ^ε M, T_ε. i "" $\int\limits_{C_j^\varepsilon} f_i d\mu = f_j^\varepsilon$.

() i "" T_ε

$$F_i(\psi) = \lim_{m \to \infty} \frac{1}{m} \sum_{m=0}^{\infty} \int\limits_{T_\varepsilon^m (C_{j_0}^\varepsilon)} f_i(T_\varepsilon^m (C_{j_0}^\varepsilon)) d\mu \qquad (*)$$

2. M, T, ξ, $\varepsilon > 0$ T_ε (*) ,

$$F_i(\psi) = \int\limits_M f_i d\mu.$$

. ξ.

1. Γ I, f_i, P ε_i .

. . :

2. ( ) $\varepsilon > 0$ l, l ε-, $\varepsilon > 0$.